The Forcing Steiner Chromatic Number of a Graph

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ABSTRACT

Let $W \subseteq V(G)$ be a χ_s - set of G and let $G = (V, E)$ be a connected graph. If W is the only χ_s - set that contains T, then a subset $T \subseteq W$ is said to be a forcing subset of W. A minimum forcing subset of W is a forcing subset for W of minimum cardinality. The cardinality of a minimum forcing subset of W is represented by the forcing Steiner chromatic number of W, indicated by $f_{\chi_s}(W)$. The forcing Steiner chromatic number of G is represented by the symbol $f_{\chi_s}(G)$, and it is equal to $f_{\chi_s}(G) = min\{f_{\chi_s}(W)\}$, where the minimum is calculated across all χ_s - sets of G. Several common graphs forcing Steiner chromatic numbers are identified. These notions are examined for certain general properties. We characterise connected graphs with the forcing Steiner chromatic number of 0 *or* 1. *It is demonstrated that there exists a connected graph G such that $f_{\chi_s}(G) = a$ and $\chi_s(G) = b$, where $\chi_s(G)$ is the Steiner chromatic number of G. This is true for all numbers a, b with $0 \leq a \leq b, b \geq 2$, and $b \geq a + 2$.*

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1. Introduction

Let $G = (V, E)$ be a graph having a vertex set $V(G)$ and an edge set $E(G)$ ($V(G)$ or $E(G)$ correspondingly). In addition, we state that a graph G has size $m = |E(G)|$ and order $n = |V(G)|$. We refer to [1] for the fundamental terms used in graph theory. If and only if an edge $e = uv \in E(G)$ exists, a vertex v is next to a vertex u. If $e = uv \in E(G)$, then u is neighbour, and the set of v is neighbours is denoted by $N_G(v)$. The degree of a vertex $v \in V$ is $\deg_G(v) = |N_G(v)|$. If $\deg_G(v) = n - 1$, a vertex v is said to be a universal vertex. The *induced subgraph* $G[S]$ is the largest subgraph of G with the given vertex set S for any set S of vertices of G. If the subgraph induced by vertex v is finished, then vertex v is said to be an *extreme vertex*.

The length of the shortest $u - v$ path in a connected graph G is given by the *distance* $d(u, v)$ between two vertices u and v. With a nonempty set W of vertices in a connected graph G, the Steiner distance $d(W)$ of W is the minimum size of a connected subgraph of G containing W. In [2], the Steiner distance was investigated. Let $S(W)$ be the collection of all

Steiner W- tree vertices. A set $W \subseteq V(G)$ is referred to as a *Steiner set* of G if $S(W) = V(G)$. The lowest cardinality for a Steiner set, commonly referred to as a *minimum Steiner set* or simply an s- set, is the Steiner number $s(G)$ of G. The *Steiner number* $s(G)$ of G determines the minimal cardinality of a Steiner set, which is also known as an s - set. In the event where $G[W]$ is connected, $d(W) = |W| - 1$ and $S(W) = W$. The Steiner number concept was covered in [3-16].

A *k-coloring* of G is a function with the form $c: V(G) \to \{1, 2, \ldots, k\}$, where $c(u) \neq c(v)$ for any two adjacent vertices u and v in G. The vertices of G are given p colours, $1, 2, \ldots, p$, and the colouring is considered to be correct if no two clearly neighbouring vertices share the same colour. The *chromatic number* of G, represented by, $\chi(G)$ is the bare minimum of colours required to colour the vertices of G. G is said to as being p - chromatic if $\chi(G) = p$, where $p \leq k$. If C contains each of the unique colour vertices in G, then the set $C \subseteq V(G)$ is referred to as a *chromatic set*. The lowest cardinality among all the chromatic sets of G is the chromatic number. That is, $\chi(G) = min\{|C_i| / C_i \text{ is a chromatic set of } G\}$ denotes a chromatic set of G. In [17, 18], the chromatic number notion was explored.

If W is both a Steiner set and a chromatic set of G, it is referred to $W \subseteq V(G)$ as a Steiner chromatic set of G. The Steiner chromatic number of G, which is represented by the symbol $\chi_s(G)$, is the minimal cardinality of a Steiner chromatic set of G. [19] investigated the Steiner chromatic number theory.

In [20], the forcing notion was first discussed and introduced. Further research is found in [21, 22, 23, 24, 25]. Several authors have investigated the forcing notion in relation to various factors, including geodetic, Steiner, hull, diversion, monophonic, etc. The forcing idea in relation to minimum Stiener chromatic sets is examined in this article. The sequel makes use of the following theorems.

Theorem:1.1[19]. Every Steiner chromatic set of a connected graph G contains an extreme vertex that belongs to that set.

Theorem:1.2[19]. The Steiner chromatic set of G includes each universal vertex in the connected graph G.

Theorem:1.3[19]. For the graph $G = K_{1,a}$ $(a \geq 2)$, $\chi_s(G) = a$

2. The Forcing Steiner Chromatic Number of a Graph

Definition:2.1. Let $W \subseteq V(G)$ be a $\chi_s(G)$ set of G and let $G = (V, E)$ be a connected graph. If W is the only χ_s - set that contains T, then a subset $T \subseteq W$ is said to be a forcing subset of W. A minimum forcing subset of W is a forcing subset for $W$of minimum cardinality. The cardinality of a minimum forcing subset of W is represented by the forcing Steiner chromatic number of W, indicated by $f_{\chi_s}(W)$. The forcing Steiner chromatic number of G is denoted by $f_{\chi_s}(G)$, and it is equal to $f_{\chi_s}(G) = \min\{f_{\chi_s}(W)\}$, where the minimum is calculated across all χ_s - sets of G.

Example:2.2. For the graph G given in Figure. 2.1, assign the colors as follows:

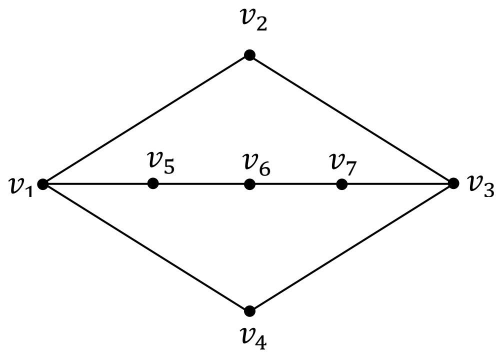


Figure: 2.1

Let $c(v_1) = 1, c(v_3) = c(v_5) = 2, c(v_2) = c(v_6) = 3$, and $c(v_4) = c(v_7) = 4$.

Then $W_1 = \{v_1, v_3, v_6, v_7\}$ and $W_2 = \{v_1, v_2, v_4, v_6\}$ are the only two χ_s - sets of G such that $f_{\chi_s}(W_1) = f_{\chi_s}(W_2) = 1$ so that $f_{\chi_s}(G) = 1$.

Definition:2.3. A vertex v is said to be a *Steiner chromatic vertex* of G if v belongs to every χ_s - sets of G.

Remark:2.4. For the graph G given in Figure. 2.1, $\{v_1, v_6\}$ is the set of all Steiner chromatic vertices of G.

Observation:2.5. Let G be a connected graph. Then

(a) for any connected graph G, $0 \leq f_{\chi_s}(G) \leq \chi_s(G)$.

(b) $f_{\chi_s}(G) = 0$ if and only if G has a distinct Steiner chromatic set with a minimum.

(c) $f_{\chi_s}(G) = 1$ if and only if G has at least two minimal Steiner chromatic sets, at least one of which is a special minimum Steiner chromatic set containing one of its components.

(d) If and only if no minimum Steiner chromatic set of G is the only minimum Steiner chromatic set that contains any of its appropriate subsets, then $f_{\chi_s}(G) = \chi_s(G)$.

(e) The set of all the Steiner chromatic vertices of G is represented by the expression $f_{\chi_s}(G) \leq \chi_s(G) - |\chi|$.

3. Some Results on Forcing Steiner Chromatic Number of G.

The forcing chromatic number of a few common graphs is determined in the section that follows.

Theorem:3.1. For a complete graph $G = K_n$ $(n \geq 2), f_{\chi_s} = 0$.

Proof: By Theorem 1.1, $W = V(G)$ is the only set of χ_s for G. The conclusion then arises from Observation 2.5(b).

Theorem:3.2. $f_{\chi_s} = 0$ if G is a connected graph with at least one universal vertex.

Proof: This follows from Theorem 1.2 and Observation 2.5(b).

Corollary:3.3. Let G to be either a fan graph F_n or a wheel graph W_n. Therefore $f_{\chi_s}(G) = 0$.

Corollary:3.4 For the graph $G = K_n - e$ $(n \geq 4), f_{\chi_s}(G) = 0$.

Corollary:3.5 For the graph $G = K_1 + \cup\, m_j k_j$ where $\sum m_j \geq 2$, $f_{\chi_s}(G) = 0$.

Theorem:3.6 For the path $G = P_n, (n \geq 4)$, $f_{\chi_s}(G) = \begin{cases} 0 & if\ n\ is\ even \\ 1 & if\ n\ is\ odd \end{cases}$.

Proof: Let $V(P_n) = \{v_1, v_2, \ldots \ldots, v_n\}$. We have the following two cases.

Case (i): n is even. Let $n = 2k$ $(k \geq 2)$. Assign the coloring as follows $c(v_{2i-1}) = c_1$ and $c(v_{2i}) = c_2, 1 \leq \mathrm{i} \leq \mathrm{k}$. Then $W = \{v_1, v_n\}$ is the unique χ_s - set of G so that $f_{\chi_s}(G) = 0$.

Case (ii): n is odd. Let $n = 2k + 1$ $(k \geq 2)$. Assign the coloring as follows

$c(v_{2i-1}) = 1$, for $1 \leq i \leq k - 1$,

$c(v_{2k+1}) = c_3$,

$c(v_{2i}) = 2$, for $1 \leq i \leq k$.

Then χ_s – set is not unique. Therefore $f_{\chi_s}(G) \geq 1$. Since $W = \{v_1, v_2, v_n\}$ is the unique χ_s - set containing $v_2, f_{\chi_s}(W) = 1$. Hence it follows that $f_{\chi_s}(G) = 1$.

Theorem:3.7. For the cycle $G = C_n$ $(n \geq 4), f_{\chi_s}(G) = \begin{cases} 0 & if\ n = 5 \\ 1 & otherwise \end{cases}$.

Proof: Let $V(C_n) = \{v_1, v_2, \ldots \ldots \ldots, v_n\}$, we have the following cases.

Case (i): n is even. Let $n = 2k$ $(k \geq 2)$, we assign coloring for each vertex as follows

$c(v_1) = c(v_3) = \ldots \ldots \ldots = c(v_{2k-1}) = c_1$ and

$c(v_2) = c(v_4) = \ldots \ldots \ldots = c(v_{2k}) = c_2$.

Then $W = \{v_1, v_{k+1}\}$ is a χ_s - set of G so that $\chi_s(G) = 2$. Since $n \geq 4$, the χ_s - set is not unique, and as a result, $f_{\chi_s}(G) \geq 1$. We have $f_{\chi_s}(G) = 1$ since W is the only χ_s - set of G that contains the value v_1.

Case (ii): n is odd. Let $n = 2k + 1$ $(k \geq 2)$. It is easily verified that no two element subset of G is not a Steiner chromatic set of G and so $\chi_s \geq 3$. Let $k = 2$. Then we assign the follows colors

$$c(v_1) = c(v_3) = c_1,$$
$$c(v_2) = c(v_5) = c_2 \text{ and}$$
$$c(v_4) = c_3.$$

Then $W = \{v_1, v_2, v_4\}$ is the unique χ_s - set of G so that $\chi_s(G) = 3$ and $f_{\chi_s}(G) = 0$. So, let $k \geq 3$. Then we assign the following colors

$$c(v_1) = c(v_3) = \ldots\ldots\ldots = c(v_k) = c(v_{k+3}) = c(v_{k+5}) = \ldots\ldots\ldots = c(v_{2k}) = 9,$$
$$c(v_2) = c(v_4) = \ldots\ldots\ldots = c(v_{k+1}) = c(v_{k+4}) = c(v_{k+6}) = \ldots\ldots\ldots = c(v_{2k+1}) = 5 \text{ and}$$
$$c(v_{k+2}) = c_3.$$

Then $W_1 = \{v_1, v_4, v_{k+2}\}$ and $W_2 = \{v_1, v_2, v_{k+2}\}$ are the only two χ_s – sets of G so that $\chi_s = 3$ and $f_{\chi_s}(G) = 1$.

Theorem:3.8. For the complete bipartite graph $G = K_{r,s}$ $(1 \leq r \leq s)$, $f_{\chi_s}(G) = 0$.

Proof: For $r = 1, s \geq 1$, by Theorem: 3.1, $f_{\chi_s}(G) = 0$. So let $2 \leq r \leq s$. Let X and Y be two bipartite sets of G. Then either X and Y is a Steiner set of G. Since for $x \in X$ and $y \in Y, xy \in E(G)$, each vertex of G is assigned by distinct colors. Then it follows that $W = V(G)$ is the unique χ_s - set of G so that $\chi_s(G) = r + s$ and $f_{\chi_s}(G) = 0$.

Theorem:3.9. For the ladder $G = K_2 \times P_n$ $(n \geq 3)$, $f_{\chi_s}(G) = \begin{cases} 1 & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$.

Proof: Let $x_1, x_2, \ldots\ldots, x_n$ and $y_1, y_2, \ldots\ldots, y_n$ be the vertices on the path P_n of the laph from the top to bottom on the left side and right side respectively. We have the following cases

Case (i): n is odd. Let $n = 2k - 1; k \geq 2$. Then assign the colors for each vertex as follows $c(x_{2i-1}) = c(y_{2i}) = 1$ for $1 \leq i \leq k$ and $c(x_{2i}) = c(y_{2i-1}) = 2$ for $1 \leq i \leq k$. Then $W_1 = \{x_1, y_{2k-1}\}$ and $W_2 = \{y_1, x_{2k-1}\}$ are the only χ_s – sets of G such that $f_{\chi_s}(W_1) = f_{\chi_s}(W_2) = 1$ so that $f_{\chi_s}(G) = 1$.

Case (ii): n is even. Let $n = 2k; k \geq 4$. Then we assign the colors for each vertex as follows.

Let $c(x_{2i-1}) = 1$ $(1 \leq i \leq k - 1)$,
$$c(y_{2i}) = 1\ (1 \leq i \leq k - 1),$$
$$c(x_{2i}) = 2\ (1 \leq i \leq k - 1),$$
$$c(y_{2i-1}) = 2\ (1 \leq i \leq k),$$
$$c(x_{2k}) = 3 \text{ and } c(y_{2k}) = 4.$$

Then $W = \{x_1, x_{2k}, y_1, y_{2k}\}$ is the unique χ_s – set of G so that $f_{\chi_s}(G) = 0$.

Theorem:3.10. Let G be the graph formed by connecting the two complete graphs $K_{r,r}$ and $K_{r,r}$ $(r \geq 2)$ along a path of any length r. Therefore $f_{\chi_s}(G) = 0$.

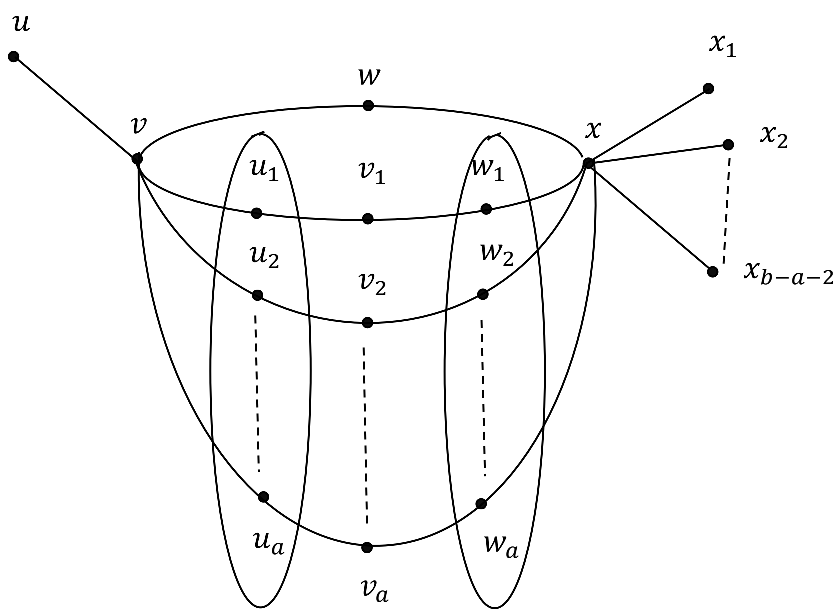
Proof: Let X and Y be the bipartite sets of first complete graph $K_{r,r}$ and, U and V be the bipartite sets of the second complete graph. Let $X = \{x_1, x_2, \dots \dots, x_r\}, Y = \{y_1, y_2, \dots \dots, y_r\}$, $U = \{u_1, u_2, \dots \dots, u_r\}$ and $V = \{v_1, v_2, \dots \dots, v_s\}$. Let $P_r : z_1, z_2, \dots \dots, z_r$. Let G be the graph obtained from $K_{r,r}, K_{r,r}$ and P_r $(r \geq 2)$ by introducing the edges $z_1 y_r$ and $z_r v_1$. We assign the colors for each vertex as follows.

$$c(x_i) = c(v_i) = c_i \ (i \leq i \leq r),$$
$$c(y_i) = c(u_i) = d_i \ (i \leq i \leq r),$$
$$c(z_1) = c_1, c(z_r) = d_1, c(z_i) = c_i \ (2 \leq i \leq r-1).$$

Then it follows that $W = \{y_1, y_2, \dots \dots, y_r\} \cup \{v_1, v_2, \dots \dots, v_s\}$ is the unique χ_s - set of G so that $f_{\chi_s}(G) = 0$.

Theorem:3.11. There is a connected graph G such that $f_{\chi_s}(G) = a$ and $\chi_s(G) = b$ for every pair of integers a, b with $0 < a < b, b \geq 2$ and $b > a + 2$.

Proof: For $a = 0$, let $G = K_{1,a}$. Then by Corollary 3.5 and Theorem 1.3, $f_{\chi_s}(G) = 0$ and $\chi_s(G) = b$. So let $a \geq 1$. Let $P: u, v, w, x$ be a path of order 4. Let $P_i: u_i, v_i, w_i$ $(1 \leq i \leq a)$ be a copy of path of order 3. Let H be the graph obtained from P and P_i $(1 \leq i \leq a)$ by the introducing the edges vu_i, xw_i, u_iu_j $(i \neq j)$ and w_iw_j $(i \neq j), (1 \leq i, j \leq a)$. Let G be the graph obtained from H by adding the new vertices $x_1, x_2, \dots \dots, x_{b-a-2}$ and introducing the edge xx_i $(1 \leq i \leq b-a-2)$. The graph G is shown in figure.



G Figure 3.1
A graph G with $f_{\chi_s}(G) = a$ and $\chi_s(G) = b$

First we prove that $\chi_s(G) = b$. Let $X = \{u, x_1, x_2, \dots \dots, x_{b-a-2}\}$ the set of all end vertices of G by Theorem 1.1, Z is a subset of every Steiner chromatic set of G. Let $H_i = \{u_i, w_i\}$ $(1 \leq i \leq a)$. Then it is easily seen that every Steiner chromatic set of G contains exactly are vertex from each $H_i (1 \leq i \leq a)$. Let us assign colors for each vertex as follows.

$c(u) = c(x) = c$, $c(v_i) = c$ $(1 \leq i \leq a)$,

$c(x_i) = c_i$ $(1 \leq i \leq b - a - 2)$,

$c(u_i) = c(v_i) = d_i$ $(1 \leq i \leq a)$,

$c(v) = c_1$ and

$c(w) = d_1$.

Let $Z = X \cup \{x\}$. Then Z is a subset of every Steiner chromatic set of G and so $\chi_s(G) \geq b - a + a = b$. Let $W = \{u_1, u_2, \dots \dots, u_a\}$. Then W is Steiner chromatic set of G so that $\chi_s(G) = b$.

Next we prove that $f_{\chi_s}(G) = a$. By Observation 2.5 (e), $f_{\chi_s}(G) \leq b - (b - a) = b$. Since Z is a subset of every Steiner chromatic set of G and every chromatic set of G contains exactly one vertex from each H_i $(i \leq i \leq a)$, every χ_s - set is of the form $W = Z \cup \{c_1, c_2, \dots \dots, c_a\}$, where $C_i \in H_i$ $(1 \leq i \leq a)$. Let T be a proper subset of W with $|T| < a$. Then for some i, $H_i \cap T = \emptyset$. This shows that $f_{\chi_s}(G) = a$.

Conclusion:

In this paper the concept of forcing Steiner chromatic number of some standard graphs some general properties satisfied by this concept are studied. In future studies, the same concept will be applied for the other graph operations.

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